

TANGENT POWER SUMS AND THEIR APPLICATIONS

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ABSTRACT. For integer m, p , we study tangent power sum $\sum_{k=1}^m \tan^{2p} \frac{\pi k}{2m+1}$. We prove that, for every m, p , it is integer, and, for a fixed p , it is a polynomial in m of degree $2p$. We give recurrent, asymptotical and explicit formulas for these polynomials and indicate their connections with Newman's digit sums in base $2m$.

1. INTRODUCTION

Recently Merca [3] found a simple formula for cosine power sum

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2p} \frac{\pi k}{n} = -\frac{1}{2} + \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn}.$$

In the present paper, for odd n , we study tangent power sum of the same form

$$(1) \quad \sigma(n, p) = \sum_{k=1}^{\frac{n-1}{2}} \tan^{2p} \frac{\pi k}{n}.$$

Everywhere below we suppose that $n \geq 1$ is an odd number and p is a positive integer. We prove the following theorems.

Theorem 1. *For every p , $\sigma(n, p)$ is integer and multiple of n .*

Theorem 2. *For a fixed p , $\sigma(n, p)$ is a polynomial in n of degree $2p$ with the leading term*

$$(2) \quad \frac{2^{2p-1}(2^{2p}-1)}{(2p)!} |B_{2p}| n^{2p},$$

where B_{2p} is Bernoulli number.

Besides, we find several other representations for $\sigma(n, p)$ (see Theorems 3 and 8) and numerical results (Section 4).

2. PROOF OF THEOREM 1

Denote $\omega = e^{\frac{2\pi i}{n}}$. Note that

$$(3) \quad \tan \frac{\pi k}{n} = i \frac{1 - \omega^k}{1 + \omega^k} = -i \frac{1 - \omega^{-k}}{1 + \omega^{-k}}, \quad \tan^2 \frac{\pi k}{n} = \frac{1 - \omega^{-k}}{1 + \omega^k} \frac{1 - \omega^k}{1 + \omega^{-k}}$$

and, for the factors of $\tan^2 \frac{\pi k}{n}$, we have

$$(4) \quad \frac{1 - \omega^{-k}}{1 + \omega^k} = \frac{(-\omega^k)^{n-1} - 1}{(-\omega^k) - 1} = \sum_{j=0}^{n-2} (-\omega^k)^j, \quad \frac{1 - \omega^k}{1 + \omega^{-k}} = \sum_{j=0}^{n-2} (-\omega^{-k})^j.$$

Since $\tan \frac{\pi k}{n} = -\tan \frac{\pi(n-k)}{n}$, then we have

$$(5) \quad 2\sigma(n, p) = \sum_{k=1}^{n-1} \tan^{2p} \frac{\pi k}{n}$$

and, by (3)-(5),

$$(6) \quad \begin{aligned} 2\sigma(n, p) &= \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-2} (-\omega^k)^j \right)^p \left(\sum_{j=0}^{n-2} (-\omega^{-k})^j \right)^p = \\ &= \sum_{k=1}^{n-1} \left(\prod_{l=0}^{p-1} \sum_{j=0}^{n-2} (-\omega^k)^{jl} \prod_{l=0}^{p-1} \sum_{j=0}^{n-2} (-\omega^{-k})^{jl} \right) = \\ &= \sum_{k=1}^{n-1} \left(\prod_{t=0}^{2p-1} \sum_{j=0}^{n-2} (-\omega^{(-1)^t k})^j \right). \end{aligned}$$

Furthermore, we note that

$$(7) \quad (n-1)^t \equiv (-1)^t \pmod{n}.$$

Indeed, it is evident for odd t . If t is even and $t = 2^h s$ with odd s , then

$$\begin{aligned} (n-1)^t - (-1)^t &= ((n-1)^s)^{2^h} - ((-1)^s)^{2^h} = \\ &= ((n-1)^s - (-1)^s)((n-1)^s + (-1)^s)((n-1)^{2s} + \\ &\quad (-1)^{2s}) \cdot \dots \cdot ((n-1)^{2^{h-1}s} + (-1)^{2^{h-1}s}), \end{aligned}$$

and, since $(n-1)^s + 1 \equiv 0 \pmod{n}$, we are done. Using (7), we can write

(6) in the form (we sum from $k = 0$, adding the zero summand)

$$(8) \quad 2\sigma(n, p) = \sum_{k=0}^{n-1} \prod_{t=0}^{2p-1} (1 - \omega^{k(n-1)^t} + \omega^{2k(n-1)^t} - \dots - \omega^{(n-2)k(n-1)^t}).$$

Considering $0, 1, 2, \dots, n-2$ as digits in the base $n-1$, after the multiplication of factors of the product in (8) we obtain summands of the form $(-1)^{s(r)} \omega^{kr}$, $r = 0, \dots, (n-1)^{2p} - 1$, where $s(r)$ is the digit sum of r in the base $n-1$. Thus we have

$$(9) \quad 2\sigma(n, p) = \sum_{k=0}^{n-1} \sum_{r=0}^{(n-1)^{2p}-1} (-1)^{s(r)} \omega^{kr} = \sum_{r=0}^{(n-1)^{2p}-1} (-1)^{s(r)} \sum_{k=0}^{n-1} (\omega^k)^r.$$

However,

$$\sum_{k=0}^{n-1} (\omega^k)^r = \begin{cases} n, & \text{if } r \equiv 0 \pmod{n} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by (9),

$$(10) \quad 2\sigma(n, p) = n \sum_{r=0, n|r}^{(n-1)^{2p}-1} (-1)^{s(r)}$$

and, consequently, $2\sigma(n, p)$ is integer multiple of n . It is left to show that the right hand side of (10) is even. It is sufficient to show that the sum contains even number of summands. The number of summands is

$$\begin{aligned} 1 + \lfloor \frac{(n-1)^{2p}}{n} \rfloor &= 1 + \frac{(n-1)^{2p} - 1}{n} = \\ 1 + \sum_{l=0}^{2p-1} (-1)^l \binom{2p}{l} n^{2p-1-l} &\equiv 1 + \sum_{l=0}^{2p-1} (-1)^l \binom{2p}{l} \pmod{2} = \\ 1 - (-1)^{2p} \binom{2p}{2p} &= 0. \end{aligned}$$

This completes proof of the theorem. \square

3. PROOF OF THEOREM 2

As is well known,

$$\sin n\alpha = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \cos^{n-(2i+1)} \alpha \sin^{2i+1} \alpha,$$

or

$$\sin n\alpha = \tan \alpha \cos^n \alpha \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \tan^{2i} \alpha.$$

Put here $\alpha = \frac{k\pi}{n}$, $k = 1, 2, \dots, \frac{n-1}{2}$. Since $\tan \alpha \neq 0$, $\cos \alpha \neq 0$, then

$$\begin{aligned} 0 &= \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} \tan^{2i} \alpha = \\ &(-1)^{\frac{n-1}{2}} (\tan^{n-1} \alpha - \binom{n}{n-2} \tan^{n-3} \alpha + \dots - \\ &(-1)^{\frac{n-1}{2}} \binom{n}{3} \tan^2 \alpha + (-1)^{\frac{n-1}{2}} \binom{n}{1}). \end{aligned}$$

This means that the equation

$$(11) \quad \lambda^{\frac{n-1}{2}} - \binom{n}{2} \lambda^{\frac{n-3}{2}} + \binom{n}{4} \lambda^{\frac{n-5}{2}} - \dots + (-1)^{\frac{n-1}{2}} \binom{n}{n-1} = 0$$

has $\frac{n-1}{2}$ roots: $\lambda_k = \tan^2 \frac{k\pi}{n}$, $k = 1, 2, \dots, \frac{n-1}{2}$. Note that (11) is the characteristic equation for the following difference equation

$$(12) \quad \begin{aligned} y(p) &= \binom{n}{2} y(p-1) - \binom{n}{4} y(p-2) + \dots - \\ &(-1)^{\frac{n-1}{2}} \binom{n}{n-1} y(p - \frac{n-1}{2}) \end{aligned}$$

which, consequently, has a private solution

$$y(p) = \sum_{k=1}^{\frac{n-1}{2}} (\tan^2 \frac{k\pi}{n})^p = \sigma(n, p).$$

Now, using Newton's formulas for equation (11),

$$\begin{aligned} \sigma(n, 1) &= \binom{n}{2}, \\ \sigma(n, 2) &= \binom{n}{2} \sigma(n, 1) - 2 \binom{n}{4}, \\ (13) \quad \sigma(n, 3) &= \binom{n}{2} \sigma(n, 2) - \binom{n}{4} \sigma(n, 1) + 3 \binom{n}{6}, \text{ etc.} \end{aligned}$$

we conclude that $\sigma(n, p)$ is a polynomial in n of degree $2p$. Note that, by induction, all these polynomials are integer-valued and thus we have another independent proof of Theorem 1. To find the leading terms of these polynomials, we make some transformations of (1). Put $\frac{n-1}{2} = m$. Changing in (1) the order of summands ($l = m - k$) and noting that

$$\frac{(m-l)\pi}{2m+1} + \frac{(2l+1)\pi}{4m+2} = \frac{\pi}{2},$$

we have

$$(14) \quad \sigma(n, p) = \sum_{l=0}^{m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2}.$$

Further we have

$$\begin{aligned} \sigma(n, p) &= \sum_{0 \leq l \leq \sqrt{m}} \cot^{2p} \frac{(2l+1)\pi}{4m+2} + \\ (15) \quad &\sum_{\sqrt{m} < l \leq m-1} \cot^{2p} \frac{(2l+1)\pi}{4m+2} = \Sigma_1 + \Sigma_2. \end{aligned}$$

Let $p > 1$. Let us estimate the second sum Σ_2 . The convexity of $\sin x$ on $[0, \frac{\pi}{2}]$ gives the inequality $\sin x \geq \frac{2}{\pi}x$. Therefore, for summands in the second sum, we have

$$\begin{aligned} \cot^{2p} \frac{(2l+1)\pi}{4m+2} &< \sin^{-2p} \frac{(2l+1)\pi}{4m+2} < \\ &(\frac{2m+1}{2l+1})^{2p} < (\frac{2m+1}{2\sqrt{m}+1})^{2p} < m^p. \end{aligned}$$

This means that $\Sigma_2 < m^{p+1} < m^{2p}$ and not influences on the leading term.

Now note that, evidently,

$$\frac{(2l+1)\pi}{4m+2} \cot \frac{(2l+1)\pi}{4m+2} \rightarrow 1$$

uniformly over $l \leq \sqrt{m}$. Thus

$$\begin{aligned}\Sigma_1 &= \sum_{0 \leq l \leq \sqrt{m}} \left(\frac{(4m+2)}{(2l+1)\pi} \right)^{2p} + \alpha(m) = \\ &= \left(\frac{(4m+2)}{\pi} \right)^{2p} \sum_{0 \leq l \leq \sqrt{m}} \frac{1}{(2l+1)^{2p}} + \alpha(m),\end{aligned}$$

where $\alpha(m) \leq \varepsilon\sqrt{m}$. Thus the coefficient of the leading term of the polynomial $\sigma(n, p)$ is

$$\begin{aligned}\lim_{m \rightarrow \infty} \frac{\Sigma_1}{n^{2p}} &= \left(\frac{2}{\pi} \right)^{2p} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^{2p}} = \\ &= \left(\frac{2}{\pi} \right)^{2p} \left(\zeta(2p) - \sum_{l=1}^{\infty} \frac{1}{(2l)^{2p}} \right) = \\ &= \left(\frac{2}{\pi} \right)^{2p} \left(\zeta(2p) - \frac{1}{2^{2p}} \zeta(2p) \right) = \frac{2^p(2^{2p}-1)}{\pi^{2p}} \zeta(2p).\end{aligned}$$

It is left to note that, by very known formula, $\zeta(2p) = \frac{|B_{2p}| 2^{2p-1} \pi^{2p}}{(2p)!}$, we find that the leading coefficient is defined by formula (2). \square

4. SEVERAL NUMERICAL RESULTS

Since, by (1), $\sigma(1, p) = 0$, then $\sigma(n, p) \equiv 0 \pmod{n(n-1)}$. Put

$$\sigma^*(n, p) = 2\sigma(n, p)/(n(n-1)).$$

By formulas (13), the first polynomials $\{\sigma^*(n, p)\}$ are

$$\begin{aligned}\sigma^*(n, 1) &= 1, \\ \sigma^*(n, 2) &= \frac{n^2 + n}{3} - 1, \\ \sigma^*(n, 3) &= \frac{2(n^2 + n)(n^2 - 4)}{15} + 1, \\ \sigma^*(n, 4) &= \frac{(n^2 + n)(17n^4 - 95n^2 + 213)}{315} - 1, \\ \sigma^*(n, 5) &= \frac{2(n^2 + n)(n^2 - 4)(31n^4 - 100n^2 + 279)}{2835} + 1, \text{ etc.}\end{aligned}$$

As well known (cf. Problem 85 in [5]), the integer-valued polynomials have integer coefficients in the binomial basis $\left\{ \binom{n}{k} \right\}$. The first integer-valued polynomials $\{\sigma(n, p)\}$ represented in binomial basis have the form

$$\begin{aligned}\sigma(n, 1) &= \binom{n}{2}, \\ \sigma(n, 2) &= \binom{n}{2} + 6\binom{n}{3} + 4\binom{n}{4}, \\ \sigma(n, 3) &= \binom{n}{2} + 24\binom{n}{3} + 96\binom{n}{4} + 120\binom{n}{5} + 48\binom{n}{6},\end{aligned}$$

$$\sigma(n, 4) = \binom{n}{2} + 78\binom{n}{3} + 836\binom{n}{4} + 3080\binom{n}{5} + 5040\binom{n}{6} + 3808\binom{n}{7} + 1088\binom{n}{8},$$

etc.

Note that the recursion (12) presupposes a fixed n . In general, by (12), we have

$$\sigma(n, p) = \binom{n}{2}\sigma(n, p-1) - \binom{n}{4}\sigma(n, p-2) + \dots - (-1)^{\frac{n-1}{2}}\binom{n}{n-1}\sigma(n, p - \frac{n-1}{2}), \quad p \geq \frac{n-1}{2}.$$

Since from (1) $\sigma(n, 0) = \frac{n-1}{2}$, $n = 3, 5, \dots$, then, calculating other initials by (13), we have the recursions:

$$\begin{aligned} \sigma(3, p) &= 3\sigma(3, p-1), \quad p \geq 1, \quad \sigma(3, 0) = 1; \\ \sigma(5, p) &= 10\sigma(5, p-1) - 5\sigma(5, p-2), \quad p \geq 2, \quad \sigma(5, 0) = 2, \quad \sigma(5, 1) = 10; \\ \sigma(7, p) &= 21\sigma(7, p-1) - 35\sigma(7, p-2) + 7\sigma(7, p-3), \quad p \geq 3, \\ \sigma(7, 0) &= 3, \quad \sigma(7, 1) = 21, \quad \sigma(7, 2) = 371; \\ \sigma(9, p) &= 36\sigma(9, p-1) - 126\sigma(9, p-2) + 84\sigma(9, p-3) - 9\sigma(9, p-4), \quad p \geq 4, \\ \sigma(9, 0) &= 4, \quad \sigma(9, 1) = 36, \quad \sigma(9, 2) = 1044, \quad \sigma(9, 3) = 33300; \text{ etc.} \end{aligned}$$

Thus

$$(17) \quad \sigma(3, p) = 3^p,$$

and a few terms of the other sequences $\{\sigma(n, p)\}$ are

$$\begin{aligned} n = 5) \quad & 2, 10, 90, 850, 8050, 76250, 722250, 6841250, 64801250, \\ & 613806250, 5814056250, \dots; \\ n = 7) \quad & 3, 21, 371, 7077, 135779, 2606261, 50028755, 960335173, \\ & 18434276035, 353858266965, 6792546291251, \dots; \\ n = 9) \quad & 4, 36, 1044, 33300, 1070244, 34420356, 1107069876, \\ & 35607151476, 1145248326468, 36835122753252, \dots; \\ n = 11) \quad & 5, 55, 2365, 113311, 5476405, 264893255, 12813875437, \\ & 619859803695, 29985188632421, 1450508002869079, \dots \end{aligned}$$

5. APPLICATIONS TO DIGIT THEORY

For $x \in \mathbb{N}$ and $n \geq 3$, denote by $S_n(x)$ the sum

$$(18) \quad S_n(x) = \sum_{0 \leq r < x: r \equiv 0 \pmod{n}} (-1)^{s_{n-1}(r)},$$

where $s_{n-1}(r)$ is the digit sum of r in base $n - 1$.

Note that, in particular, $S_3(x)$ equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [9]) in interval $[0, x)$.

Leo Moser (cf. [4], Introduction) conjectured that always

$$(19) \quad S_3(x) > 0.$$

Newman [4] proved this conjecture. Moreover, he obtained the inequalities

$$(20) \quad \frac{1}{20} < S_3(x)x^{-\lambda} < 5,$$

where

$$(21) \quad \lambda = \frac{\ln 3}{\ln 4} = 0.792481\dots$$

In connection with these remarkable Newman results, the qualitative result (19) we call a weak Newman phenomenon (or Moser-Newman phenomenon), while an estimating result of the form (20) we call a strong Newman phenomenon.

In 1983, Coquet [1] studied a very complicated continuous and nowhere differentiable fractal function $F(x)$ with period 1 for which

$$(22) \quad S_3(3x) = x^\lambda F\left(\frac{\ln x}{\ln 4}\right) + \frac{\eta(x)}{3},$$

where

$$(23) \quad \eta(x) = \begin{cases} 0, & \text{if } x \text{ is even,} \\ (-1)^{s_2(3x-1)}, & \text{if } x \text{ is odd.} \end{cases}$$

He obtained that

$$(24) \quad \limsup_{x \rightarrow \infty, x \in \mathbb{N}} S_3(3x)x^{-\lambda} = \frac{55}{3} \left(\frac{3}{65}\right)^\lambda = 1.601958421\dots,$$

$$(25) \quad \liminf_{x \rightarrow \infty, x \in \mathbb{N}} S_3(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538 \dots$$

In 2007, Shevelev [8] gave an elementary proof of Coquet's formulas (24)-(25) and his sharp estimates in the form

$$(26) \quad \frac{2\sqrt{3}}{3}x^\lambda \leq S_3(3x) \leq \frac{55}{3} \left(\frac{3}{65} \right)^\lambda x^\lambda, \quad x \in \mathbb{N}.$$

Besides, Shevelev showed that the sequence $\{(-1)^{s_2(n)}(S_3(n) - 3S_3(\lfloor n/4 \rfloor))\}$, is periodic with period 24 taking the values $-2, -1, 0, 1, 2$. This gives a simple recursion for $S_3(n)$. In 2008, Drmota and Stoll [2] proved a generalized weak Newman phenomenon, showing that (19) is valid for sum (18) for every $n \geq 3$, at least beginning with $x \geq x_0(n)$. Our proof of Theorem 1 allows to consider a strong form of this generalization, but yet only in "full" intervals in even base $n - 1$ of the form $[0, (n - 1)^{2p}]$.

Theorem 3. *For $x_{n,p} = (n - 1)^{2p}$, $p \geq 1$, we have*

$$(27) \quad S_n(x_{n,p}) \sim \frac{2}{n} x_{n,p}^\lambda \quad (p \rightarrow \infty),$$

where

$$(28) \quad \lambda = \lambda_n = \frac{\ln \cot(\frac{\pi}{2n})}{\ln(n - 1)}.$$

Proof. According to (10) and (18), we have

$$(29) \quad S_n(x_{n,p}) = \frac{2}{n} \sigma(n, p), \quad p \geq 1.$$

Thus, choosing the maximal exponent in (1) as $p \rightarrow \infty$, we find

$$(30) \quad \begin{aligned} S_n(x_{n,p}) &\sim \frac{2}{n} \tan^{2p} \frac{(n-1)\pi}{2n} = \\ &\frac{2}{n} \cot^{2p} \frac{\pi}{2n} = \exp\left(\ln \frac{2}{n} + 2p \ln \cot \frac{\pi}{2n}\right) = \\ &\exp\left(\ln \frac{2}{n} + 2p\lambda \ln(n-1)\right) = \exp\left(\ln \frac{2}{n} + \ln x_{n,p}^\lambda\right) = \frac{2}{n} x_{n,p}^\lambda. \end{aligned}$$

□

In particular, in the cases of $n = 3, 5, 7, 9, 11$ we have $\lambda_3 = \frac{\ln 3}{\ln 4} = 0.79248125\dots$, $\lambda_5 = 0.81092244\dots$, $\lambda_7 = 0.82452046\dots$, $\lambda_9 = 0.83455828\dots$, $\lambda_{11} = 0.84230667\dots$ respectively.

Show that

$$(31) \quad 1 - \frac{\ln \frac{\pi}{2}}{\ln(n-1)} \leq \lambda_n \leq 1 - \frac{\ln \frac{\pi}{2}}{\ln(n-1)} + \frac{1}{(n-1) \ln(n-1)}.$$

Indeed, by the convexity of $\cos x$ on $[0, \frac{\pi}{2}]$, $\cos x \geq 1 - \frac{2}{\pi}x$, and, therefore, $\cos \frac{\pi}{2n} \geq 1 - \frac{1}{n}$. Using also that $\tan \frac{\pi}{2n} \geq \frac{\pi}{2n} \geq \sin \frac{\pi}{2n}$, we have

$$\frac{2}{\pi}(n-1) \leq \cot \frac{\pi}{2n} \leq \frac{2}{\pi}n$$

and, by (28),

$$1 - \frac{\ln \frac{\pi}{2}}{\ln(n-1)} \leq \lambda_n \leq 1 - \frac{\ln \frac{\pi}{2}}{\ln(n-1)} + \frac{\ln(1 + \frac{1}{n-1})}{\ln(n-1)}$$

which yields (31), since, for $n \geq 3$, $\ln(1 + \frac{1}{n-1}) < \frac{1}{n-1}$. Finally, let us show the monotonic increasing of λ_n . For function $f(x) = \frac{\ln \cot(\frac{\pi}{2x})}{\ln(x-1)}$, we have

$$(32) \quad \ln(x-1)f'(x) = \frac{\pi}{x^2 \sin \frac{\pi}{x}} - \frac{f(x)}{x-1}.$$

As in (31), we also have

$$(33) \quad f(x) \leq 1 - \frac{\ln \frac{\pi}{2}}{\ln(x-1)} + \frac{1}{(x-1)\ln(x-1)}.$$

On the other hand, since $\sin \frac{\pi}{x} \leq \frac{\pi}{x}$, then

$$\frac{\pi(x-1)}{x^2 \sin \frac{\pi}{x}} \geq 1 - \frac{1}{x},$$

and, by (32), in order to show that $f'(x) > 0$, it is sufficient to prove that $f(x) < 1 - \frac{1}{x}$, or, by (33), to show that

$$1 - \frac{\ln \frac{\pi}{2}}{\ln(x-1)} + \frac{1}{(x-1)\ln(x-1)} < 1 - \frac{1}{x},$$

or

$$\frac{\ln(x-1)}{x} + \frac{1}{x-1} < \ln \frac{\pi}{2}.$$

This inequality holds for $x \geq 7$, and since $\lambda_3 < \lambda_5 < \lambda_7$, then the monotonicity of λ_n follows. Thus we have the monotonic strengthening of the strong form of Newman-like phenomenon for the base $n-1$ in the considered intervals.

In connection with the sharp estimates (26) in case $n=3$, it is natural to pose the following conjecture.

Conjecture 4. 1) Let

$$A_n = \liminf_{x \rightarrow \infty, x \in \mathbb{N}} S_n(nx)x^{-\lambda_n},$$

$$B_n = \limsup_{x \rightarrow \infty, x \in \mathbb{N}} S_n(nx)x^{-\lambda_n}.$$

Then

$$0 < A_n < B_n < \infty;$$

2) For $x \geq x_0(n)$, $x \in \mathbb{N}$,

$$A_n x^{\lambda_n} \leq S_n(nx) \leq B_n x^{\lambda_n}.$$

6. AN IDENTITY

Since (29) was proved for $x_{n,p} = (n-1)^{2p}$, $p \geq 1$, then, by (16), for $S_n(x_{n,p})$ in the case $p \geq \frac{n+1}{2}$, we have the relation

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k} S_n((n-1)^{2p-2k}) = 0.$$

In case $p = \frac{n-1}{2}$ this relation does not hold. Let us show that in this case we have the identity

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k} S_n((n-1)^{n-2k-1}) = (-1)^n,$$

or, putting $n-2k-1 = 2j$, the identity

$$(34) \quad \sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_n((n-1)^{2j}) = 1.$$

Indeed, in case $j = 0$, we, evidently, have $S_n(1) = 1$, while, formally, by (29), for $p = 0$, we obtain " $S_n(1) = \frac{2}{n} \sigma(n, 0) = \frac{2}{n} \frac{n-1}{2} = \frac{n-1}{n}$ ", i.e., the error is $-\frac{1}{n}$, and the error in the corresponding sum is $n(-\frac{1}{n}) = -1$. Therefore, in the latter formula, instead of 0, we have 1.

7. EXPLICIT COMBINATORIAL REPRESENTATION

In its turn, the representation (29) allows to get an explicit combinatorial representation for $\sigma(n, p)$. We need three lemmas.

Lemma 5. ([7], p. 215) *The number of compositions $C(m, n, s)$ of m with n positive parts not exceeding s is given by formula*

$$(35) \quad C(m, n, s) = \sum_{j=0}^{\min(n, \lfloor \frac{m-n}{s} \rfloor)} (-1)^j \binom{n}{j} \binom{m-sj-1}{n-1}.$$

Since, evidently, $C(m, n, 1) = \delta_{m,n}$, then, as a corollary, we have the identity

$$(36) \quad \sum_{j=0}^{\min(n, m-n)} (-1)^j \binom{n}{j} \binom{m-j-1}{n-1} = \delta_{m,n}.$$

Lemma 6. *The number of compositions $C_0(m, n, s)$ of m with n nonnegative parts not exceeding s is given by formula*

$$(37) \quad C_0(m, n, s) = \begin{cases} C(m+n, n, s+1), & \text{if } m \geq n \geq 1, s \geq 2, \\ \sum_{\nu=1}^m C(m, \nu, s) \binom{n}{n-\nu}, & \text{if } 1 \leq m < n, s \geq 2, \\ 1, & \text{if } m = 0, n \geq 1, s \geq 0, \\ 0, & \text{if } m > n \geq 1, s = 1, \\ \binom{n}{m}, & \text{if } 1 \leq m \leq n, s = 1. \end{cases}$$

Proof. Let firstly $s \geq 2$, $m \geq n \geq 1$. If to diminish on 1 every part of a composition of $m+n$ with n positive parts not exceeding $s+1$, then we obtain a composition of m with n nonnegative parts not exceeding s , such that zero parts allowed. Let, further, $s \geq 2$, $1 \leq m < n$. Consider $C(m, \nu, s)$ compositions of m with $\nu \leq m$ parts. To obtain n parts, consider $n-\nu$ zero parts, which we choose in $\binom{n}{n-\nu}$ ways. The summing over $1 \leq \nu \leq m$ gives the required result. Other cases are evident. \square

Let now $(n-1)^h \leq N < (n-1)^{h+1}$, $n \geq 3$. Consider the representation of N in the base $n-1$:

$$N = g_h(n-1)^h + \dots + g_1(n-1) + g_0,$$

where $g_i = g_i(N)$, $i = 0, \dots, h$, are digits of N , $0 \leq g_i \leq n-2$. Let

$$s^e(N) = \sum_{i \text{ is even}} g_i, \quad s^o(N) = \sum_{i \text{ is odd}} g_i.$$

Lemma 7. N is multiple of n if and only if $s^o(N) \equiv s^e(N) \pmod{n}$.

Proof. The lemma follows from the evident relation $(n-1)^i \equiv (-1)^i \pmod{n}$, $i \geq 0$. \square

Now we obtain a combinatorial explicit formula for $\sigma(n, p)$.

Theorem 8. For $n \geq 3$, $p \geq 1$, we have

$$(38) \quad \sigma(n, p) = \frac{n}{2} \sum_{j=0}^{(n-2)p} ((C_0(j, p, n-2))^2 + \sum_{k=1}^{\lfloor \frac{(n-2)p-j}{n} \rfloor} (-1)^k C_0(j, p, n-2) C_0(j+nk, p, n-2)),$$

where $C_0(m, n, s)$ is defined by formula (37).

Proof. Consider all nonnegative integers N 's not exceeding $(n-1)^{2p} - 1$, which have $2p$ digits $g_i(N)$ in base $n-1$ (the first 0's allowed). Let the sum of digits of N on even p positions be j , while on odd p positions such sum be $j + kn$ with a positive integer k . Then, by Lemma 7, such N 's are

multiple of n . Since in the base $n-1$ the digits not exceed $n-2$, then the number of ways to choose such $N's$, for $k=0$, is $(C_0(j, p, n-2))^2$. In the case $k \geq 1$, we should also consider the symmetric case when on odd p positions the sum of digits of N be j , while on even p positions such sum be $j+kn$ with a positive integer k . This, for $k \geq 1$, gives $2C_0(j, p, n-2)C_0(j+kn, p, n-2)$ required numbers $N's$. Furthermore, since n is odd, then, if k is odd, then $s_{n-1}(N)$ is odd, while, if k is even, then $s_{n-1}(N)$ is even. Thus the difference $S_n((n-1)^{2p})$ between n -multiple $N's$ with even and odd digit sums equals

$$S_n((n-1)^{2p}) = \sum_j ((C_0(j, p, n-2))^2 + 2 \sum_k (-1)^k C_0(j, p, n-2) C_0(j+kn, p, n-2)).$$

Now to obtain (38), note that $0 \leq j \leq (n-2)p$, and, for $k \geq 1$, also $j+kn \leq (n-2)p$, such that $1 \leq k \leq \frac{(n-2)p-j}{n}$, and that, by (29), $\sigma(n, p) = \frac{n}{2} S_n((n-1)^{2p})$. \square

Example 9. Let $n=5, p=2$. By Theorem 8, we have

$$\begin{aligned} \sigma(5, 2) &= 2.5 \sum_{j=0}^6 ((C_0(j, 2, 3))^2 + \\ (39) \quad & 2 \sum_{k=1}^{\lfloor \frac{6-j}{3} \rfloor} (-1)^k C_0(j, 2, 3) C_0(j+5k, 2, 3)). \end{aligned}$$

We have

$$\begin{aligned} C_0(0, 2, 3) &= 1, C_0(1, 2, 3) = 2, C_0(2, 2, 3) = 3, \\ C_0(3, 2, 3) &= 4, C_0(4, 2, 3) = 3, C_0(5, 2, 3) = 2, C_0(6, 2, 3) = 1. \end{aligned}$$

Thus

$$\sum_{j=0}^6 ((C_0(j, 2, 3))^2 = 44.$$

In the cases $j=0, k=1$ and $j=1, k=1$ we have

$$C_0(0, 2, 3)C_0(5, 2, 3) = 2, \quad C_0(1, 2, 3)C_0(6, 2, 3) = 2.$$

Thus

$$2 \sum_{j=0}^6 \sum_{k=1}^{\lfloor \frac{6-j}{3} \rfloor} (-1)^k C_0(j, 2, 3) C_0(j+5k, 2, 3) = -8$$

and, by (39), we have

$$\sigma(5, 2) = 2.5(44 - 8) = 90.$$

On the other hand, by (1), we directly have

$$\sigma(5, 2) = \sum_{k=1}^2 \tan^4 \frac{\pi k}{5} = 0.278640... + 89.721359... = 89.999999...$$

Example 10. In case $n = 3$, by Theorem 8 and formulas (17), (37), we have

$$\begin{aligned} 3^p &= \frac{3}{2} \sum_{j=0}^p ((C_0(j, p, 1))^2 + \\ &\quad \sum_{k=1}^{\lfloor \frac{p-j}{3} \rfloor} (-1)^k C_0(j, p, 1) C_0(j+3k, p, 1)) = \\ &\quad \frac{3}{2} \sum_{j=0}^p \binom{p}{j}^2 + 2 \sum_{k=1}^{\lfloor \frac{p-j}{3} \rfloor} (-1)^k \binom{p}{j} \binom{p}{3k+j}. \end{aligned}$$

Thus, using well known formula $\sum_{j=0}^p \binom{p}{j}^2 = \binom{2p}{p}$, we find the identity

$$\sum_{j=0}^p \sum_{k=1}^{\lfloor \frac{p-j}{3} \rfloor} (-1)^k \binom{p}{j} \binom{p}{3k+j} = 3^{p-1} - \frac{1}{2} \binom{2p}{p},$$

or, changing the order of summing,

$$\sum_{k=1}^{\lfloor \frac{p}{3} \rfloor} (-1)^k \sum_{j=0}^{p-3k} \binom{p}{j} \binom{p}{3k+j} = 3^{p-1} - \frac{1}{2} \binom{2p}{p}.$$

Since (cf.[6],p.8)

$$(40) \quad \sum_{j=0}^{p-3k} \binom{p}{j} \binom{p}{3k+j} = \binom{2p}{p+3k},$$

then we obtain a (new) identity

$$(41) \quad \sum_{k=1}^{\lfloor \frac{p}{3} \rfloor} (-1)^{k-1} \binom{2p}{p+3k} = \frac{1}{2} \binom{2p}{p} - 3^{p-1}, \quad p \geq 1.$$

8. CONCLUSIVE REMARKS

It is interesting that, using a "distorted" tangent power sum

$$(42) \quad \varrho(n, p) = \sum_{k=1}^{\frac{n-1}{2}} \tan^{2p-2} \frac{\pi k}{n} \sin^2 \frac{\pi k}{n},$$

we obtain the value of $S_n((n-1)^{2p-1})$ by the formula

$$(43) \quad S_n((n-1)^{2p-1}) = \frac{4}{n} \varrho(n, p).$$

The latter formula could be proven in a close way which led us to (29). It

is interesting also to describe the case of even n . In addition, note (private communication of T. Amdeberhan) that our tangent power sum $\sigma(n, p)$ is also cotangent power sum $\sum_{k=1}^{\frac{n-1}{2}} \cot^{2p} \frac{(2k-1)\pi}{2n}$. It easily follows from shift formulas.

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